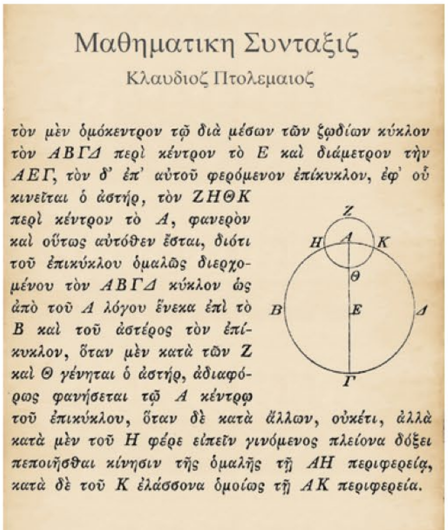
Inscribed and central angles

7 Unit



Μαθηματικη Συνταξιζ

Κλαυδιοζ Πτολεμαιοζ

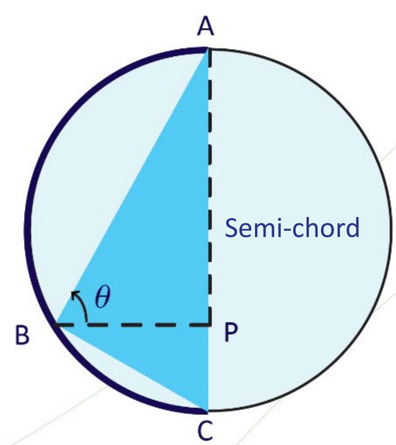
τὸν μὲν ὁμόκεντρον τῷ διὰ μέσων τῶν ζῳδίων κύκλον τὸν ΑΒΓΔ περὶ κέντρον τὸ Ε καὶ διάμετρον τὴν ΑΕΓ, τὸν δ' ἐπ' αὐτοῦ φερόμενον ἐπίκυκλον, ἐφ' οὗ κινεῖται ὁ ἀστήρ, τὸν ΖΗΘΚ περὶ κέντρον τὸ Α, φανερὸν καὶ οὕτως αὐτόθεν ἔσται, διότι τοῦ ἐπικύκλου ὁμαλῶς διερχομένου τὸν ΑΒΓΔ κύκλον ὡς ἀπὸ τοῦ Α λόγου ἕνεκα ἐπὶ τὸ Β καὶ τοῦ ἀστέρος τὸν ἐπίκυκλον, ὅταν μὲν κατὰ τῶν Ζ καὶ Θ γένηται ὁ ἀστήρ, ἀδιαφόρως φανήσεται τῷ Α κέντρῳ τοῦ ἐπικύκλου, ὅταν δὲ κατὰ ἄλλων, οὐκέτι, ἀλλὰ κατὰ μὲν τοῦ Η φέρε εἰπεῖν γινόμενος πλείονα δόξει πεποιῆσθαι κίνησιν τῆς ὁμαλῆς τῇ ΑΗ περιφερείᾳ, κατὰ δὲ τοῦ Κ ἐλάσσονα ὁμοίως τῇ ΑΚ περιφερείᾳ.

A sheet from the astronomical treatise Almagest.

Ancient civilizations used astronomy to predict abundant hunting, planting, or the arrival of winter.

In the astronomical treatise *Almagest*, the Greco-Egyptian mathematician Claudius Ptolemy (second century) made a mathematical description of the geocentric system (the planets revolve around the Earth). One of his contributions to mathematics was a theorem on cyclic quadrilaterals, in which essential properties of inscribed angles are used.

The Trigonometry, which studies the relationship between the sides and angles of a triangle, was developed by astronomical studies. During the V and VI centuries, the Indian mathematicians Varahamihira and Brahmagupta formulated numerous trigonometric properties using the semi-chord (a triangle inscribed in the circle with one side as the circle's diameter). Furthermore, the cyclic quadrilaterals are based on the study of the inscribed angles



The angle inscribed ABC is straight This construction allowed the collection of important relationships.

The contents will be developed by addressing the definition of the theorem of the inscribed angle, which establishes a relationship with the central angle. Also, study the construction of tangent lines on the circumference, the definition of semi-inscribed angles, and the relationship between chords and arcs.

1.1 Elements of the circumference

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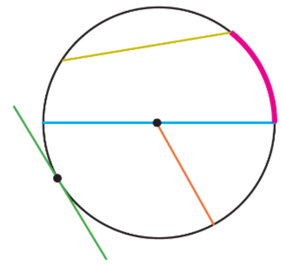
Write the name given to the drawn elements on the following circumference:



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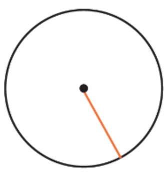
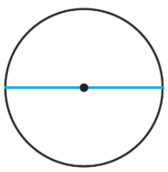
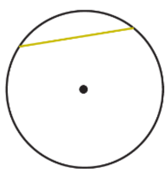
Segments.



The segment that goes from the center to a point in the circumference is called **radius.**

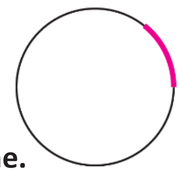


The segment that goes from one point of the circumference to another and passes through the center is called **diameter.**



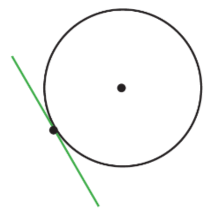
The segment that goes from one point of the circumference to another is called **chord.**

Arc.



Any portion of the circumference of a circle is called an **arc**.

Line.



The line that touches the circumference at a point is called **tangent.**

The point where the tangent line touches the circumference is called: **point of tangency**.

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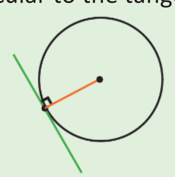
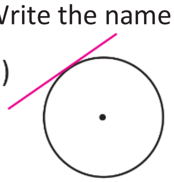
The elements of the circumference are:

- The segments: radius, diameter and chord
- The lines: tangent
- The arc of the circumference

The radius to the point of tagency is perpendicular to the tangent point.



1. Write the name of the elements given fore each circimference:

a) 

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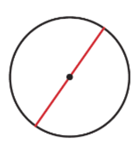
c) 

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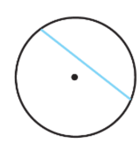
e) 

2. Respond the following questions:

   a) What is the element that is $\frac{1}{2}$ in diameter?

   b) What is the name of the longest chord on a circumference?

   c) How is the tangent line and radiues to the poingt of tangency of a circumference?

   d) By placing two dots on the circumference. How many arcs are formed?

1.2 Definition and measurement of inscribed angles

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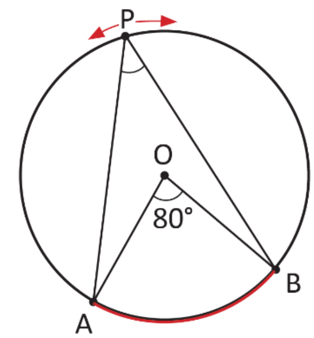
Draw on a piece of paper and measure ∢BPA by moving point P to different places in the circumference. Compare the measument of ∢BPA with ∢BOA.



The BOA angle is called **central angle**, because its vertex is the center of the circumference.

Note that ∢BPA and ∢BOA share the same arc $\overset{\frown}{AB}$.

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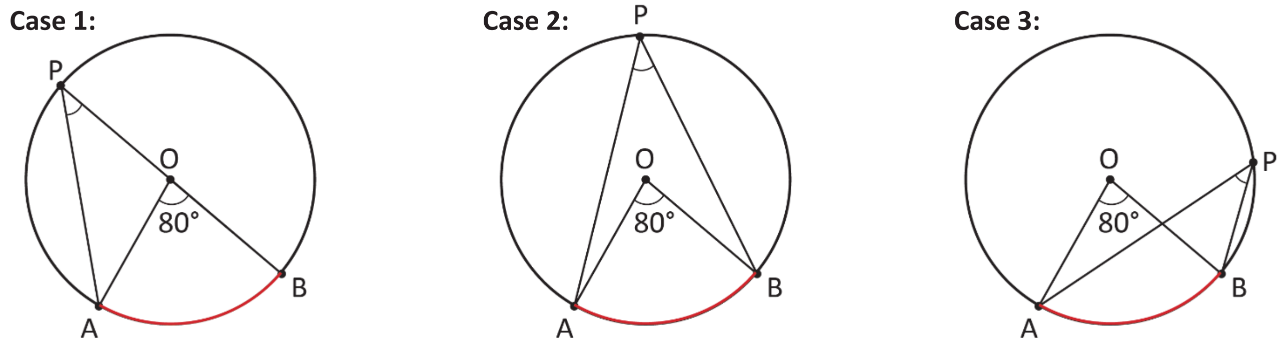
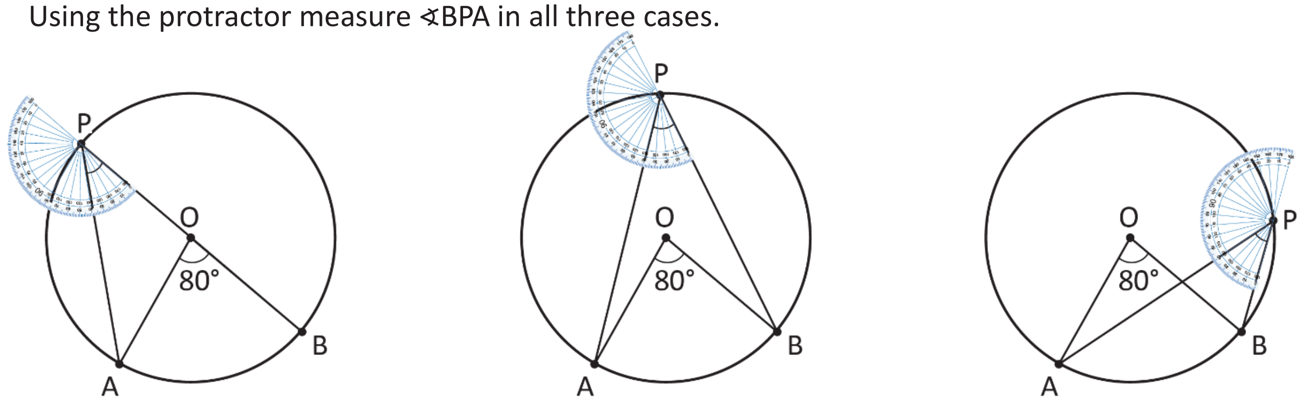
Use a ruler and a compass to make the drawing and move point P on the circumference for the following cases:

Case 1: **Case 2:** **Case 3:**



Using the protractor measure ∢BPA in all three cases.



In all three cases the measurement of ∢BPA = 40°.

And ∢BOA = 2∢BPA or ∢BPA = $\frac{1}{2}$ ∢BOA.

C

Angles whose vertex is on the circumference are called:**Inscribed angles.**

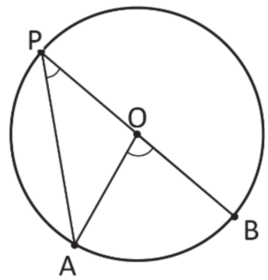
Subtending the same arc means sharing the same arc.

In a circumference, the measure of the central angle that subtends the same arc of any inscribed angle is twice the measure of any inscribed angle that subtends the same arc.

Determine the measurement of an angle inscribed to a circumference whose central angle within the same arc measure 160°. Use a scheme as in the initial problem.

1.3 Inscribed angles, part 1

Demonstrate that ∢BOA = 2∢BPA when the center lies somewhere in the ΔBPA.



The diameter is the chord that passes across the center of the circumference.

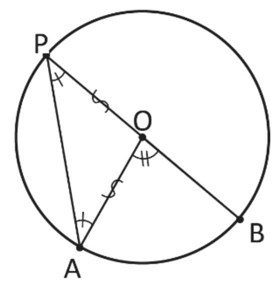
In ΔAOP: OP = OA (are radiuses of the circumference).

So, ∢OPA = ∢PAO (equal sides oposse equal angles).

Else ∢BOA = ∢OPA + ∢PAO (∢BOA is the external angle of ΔAOP).

Therefore, ∢BOA = 2∢OPA.   As ∢OPA = ∢BPA.

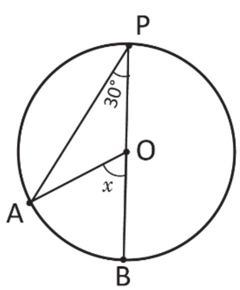
Then, ∢BOA = 2∢BPA.



In the inscribed angles whose side coincides with the diameter of the circumference it is satisfied that **the measurement of the central angle subtending the same arc is twice the measurement of the inscribed angle.**

Determine the value of x for each case.

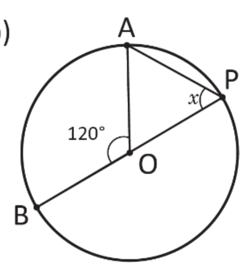
a)



As ∢BOA = 2∢BPA.

Therefore, $x = 2(30°) = 60°$.

b)



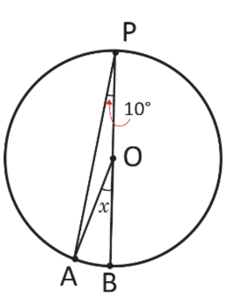
As ∢BOA = 2∢BPA.

Then, $∢BPA = \frac{1}{2}∢BOA$

Therefore, $x = \frac{120°}{2} = 60°$.

Determine the value of x for each case.

a)



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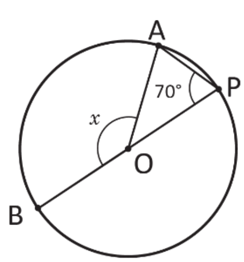

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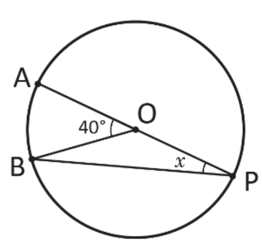

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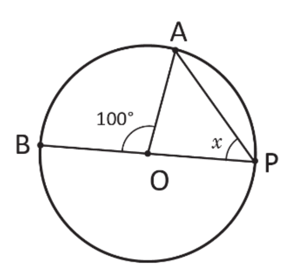

1.4 Inscribed angles, part 2

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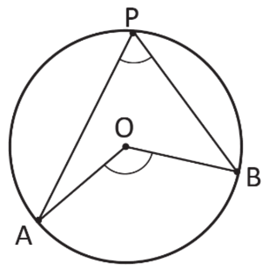
Show that ∢BOA = 2∢BPA when the center is within ∢BPA.



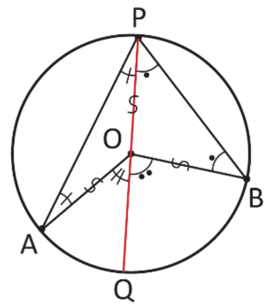
S

Draw the diameter for QP.

∢QOA = 2∢QPA and ∢BOQ = 2∢BPO (as seen in class 3).

Adding both equalities
∢QOA + ∢BOQ = 2∢QPA + 2∢BPQ = 2(∢QPA + ∢BPQ).

Therefore, ∢BOA = 2∢BPA.



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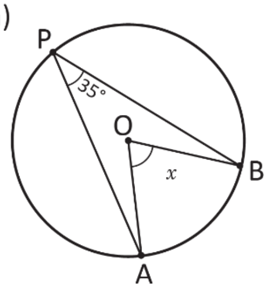
In the inscribe angles within the central angle, which subtends the same arc, comply that **the central angle measure is twice the measurement of the inscribed angle.**

E

Determine the value of x for each case.

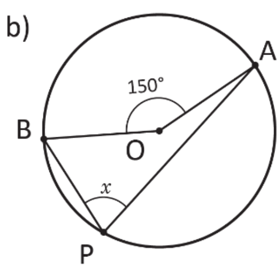
a)



As ∢BOA = 2∢BPA.

Therefore, x = 2(35°) = 70°.

b)



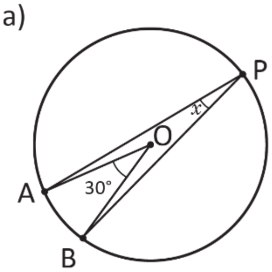
As ∢BOA = 2∢BPA.

Then, ∢BPA = $\frac{1}{2}$∢BOA.

Therefore, $x = \frac{150°}{2} = 75°$.

Determine the value of x for each case.

a)



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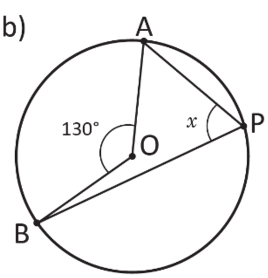

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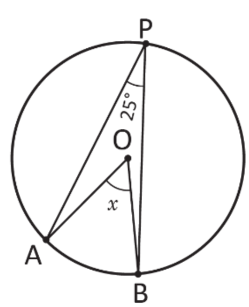

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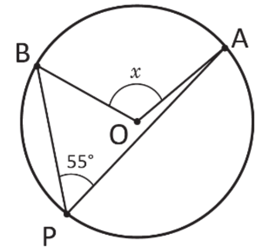

1.5 Inscribed angle theorem

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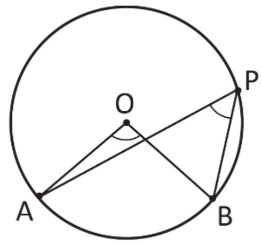
Demonstrate that ∢BOA = 2∢BPA when the center is outside of ∢BPA.



S

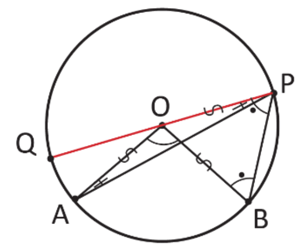
Draw the diameter for QP.

∢AOQ = 2∢APQ and ∢BOQ = 2∢BPQ (as seen in class 3).

As ∢BOA = ∢BOQ – ∢AOQ.

Then, ∢BOA = 2∢BPQ – 2∢APQ = 2(∢BPQ – ∢APQ) = 2∢BPA.

Therefore, ∢BOA = 2∢BPA.



C

In a circumference, for any inscribed angle, it is true to state that **the central angle measure is twice the measure of the inscribed angle that subtends the same arc.**

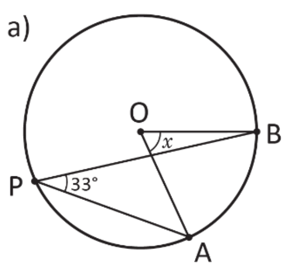
Also the inscribed angles that subtend the same arc have the equal measure.

The result is known as the **Inscribed angle theorem.**

E

Determine the value of x for each case.

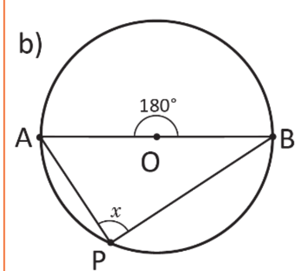
a)



As ∢BOA = 2∢BPA.

Therefore, x = 2(33°) = 66°.

b)



As ∢BOA = 2∢BPA.

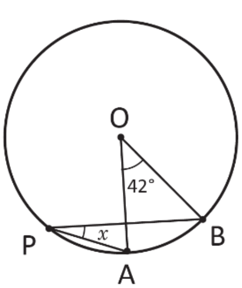
Then, ∢BPA = $\frac{1}{2}$ ∢BOA.

Therefore, $x = \frac{180°}{2} = 90°$.

The inscribed angle to the semi-circumference measures 90°.

Determine the value of x, y and z for each case.

a)



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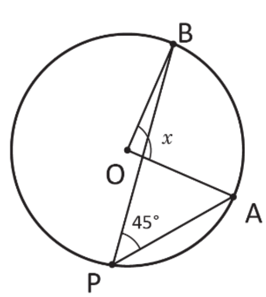

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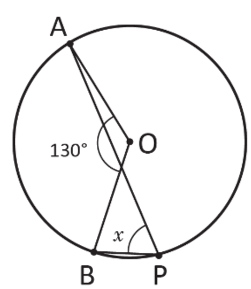

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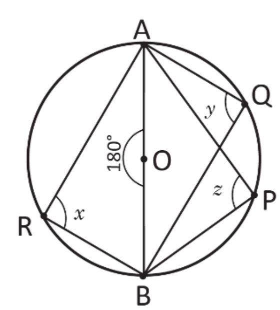
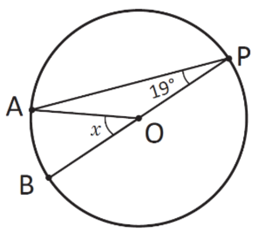


1.6 Practice what you learned

1. Determine the value of x for each case.

a)



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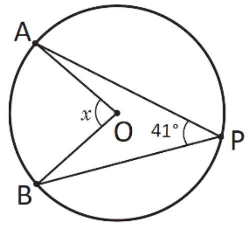

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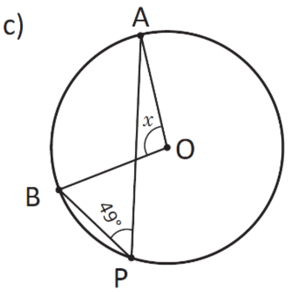

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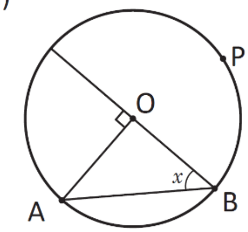

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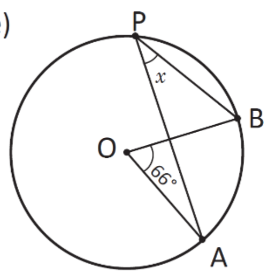

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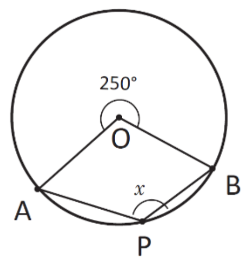

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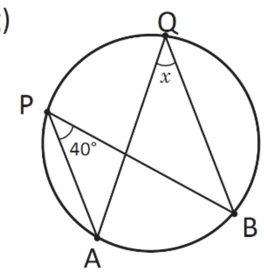

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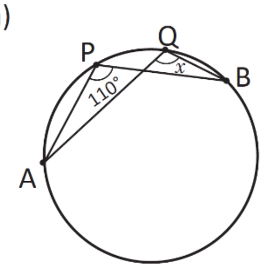
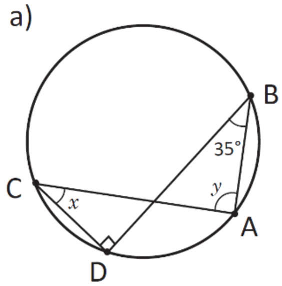


2. Determine the value of x and y according to each case.

a)



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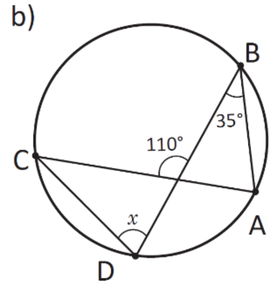

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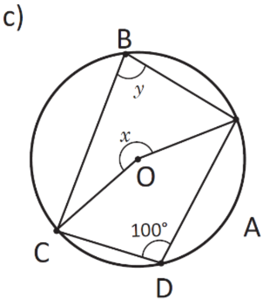

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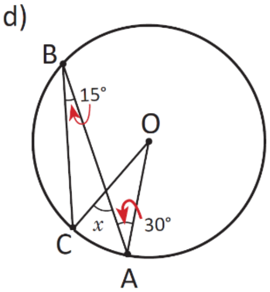

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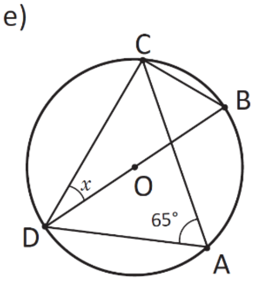

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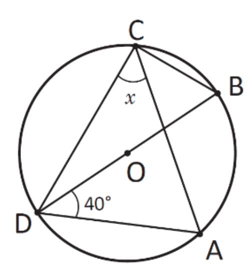

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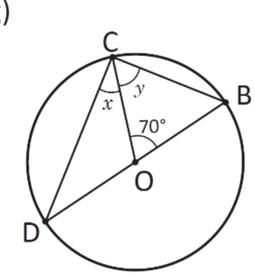
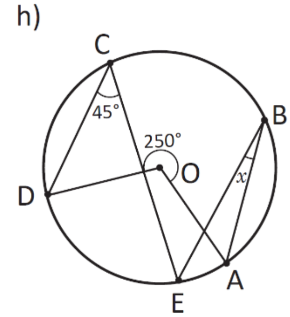


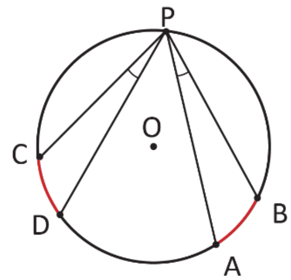
h)



1.7 Congruent arcs

P

Compare the measurement of $\sphericalangle BPA$ with $\sphericalangle DPC$ in the following figure, if $\overset{\frown}{CD} = \overset{\frown}{AB}$.



The notation $\overset{\frown}{AB}$, means the portion of the arc between point A and point B.

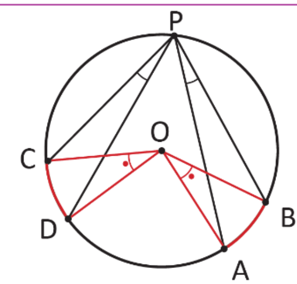
S

Construct the angles $\sphericalangle BOA$ and $\sphericalangle DOC$.

$\sphericalangle BOA = \sphericalangle DOC$ ($\overset{\frown}{CD} = \overset{\frown}{AB}$ per hypothesis).

$\sphericalangle BPA = \frac{1}{2} \sphericalangle BOA$ and $\sphericalangle DPC = \frac{1}{2} \sphericalangle DOC$ (as per inscribed angle).

Therefore, $\sphericalangle BPA = \sphericalangle DPC$.



C

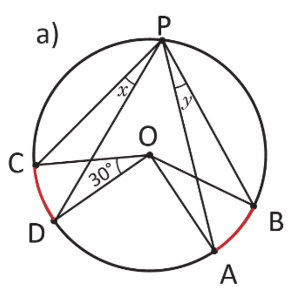
In a circumference, the inscribed angles, which subtend arcs of equal measure, have equal measure.

It is also true that if two inscribed angles are of equal measure, then the arcs that they subtend are also of equal measure.

E

Determine the value of x and y for each case where $\overset{\frown}{CD} = \overset{\frown}{AB}$.

a)

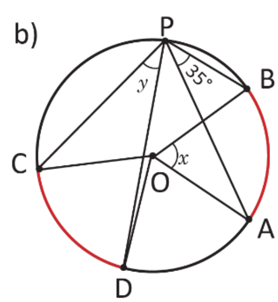


As $\sphericalangle BOA = \sphericalangle DOC$.

Therefore,

$x = y = \frac{30°}{2} = 15°$.

b)



As $\sphericalangle BOA = 2\sphericalangle BPA$.

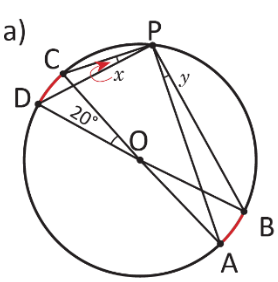
Therefore, $x = 2(35°) = 70°$.

So, $\sphericalangle BOA = \sphericalangle DOC$.

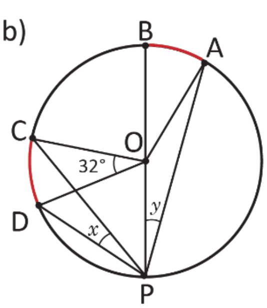
Then, $y = \sphericalangle DPC = \sphericalangle BPA = 35°$.

Determine the value of x and y for each case. Consider $\overset{\frown}{AB} = \overset{\frown}{CD}$.

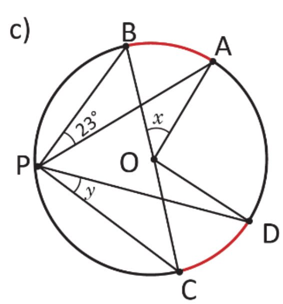
a)



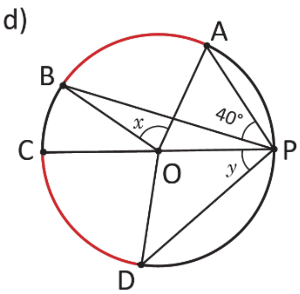
b)



c)



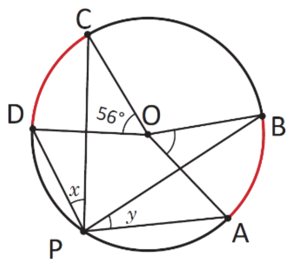
d)



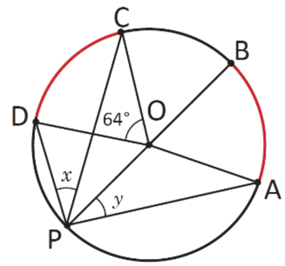
1.8 Practice what you learned

1. Determine the value of x and y for each case. Consider $\overset{\frown}{AB} = \overset{\frown}{CD}$.

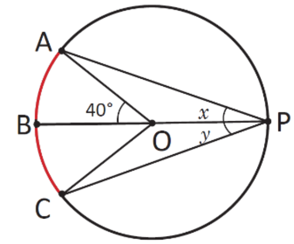
a) $\overset{\frown}{AB} = \overset{\frown}{CD}$



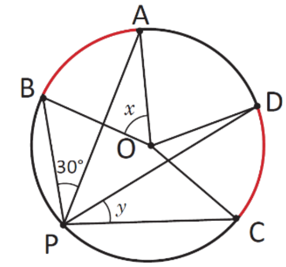
b) $\overset{\frown}{AB} = \overset{\frown}{CD}$



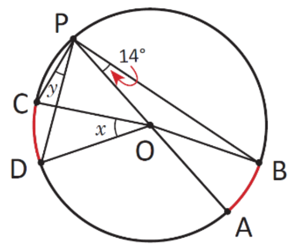
c) $\overset{\frown}{AB} = \overset{\frown}{BC}$



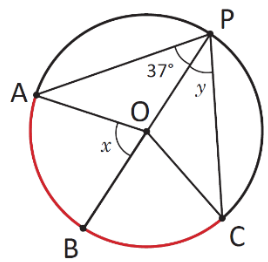
d) $\overset{\frown}{AB} = \overset{\frown}{CD}$



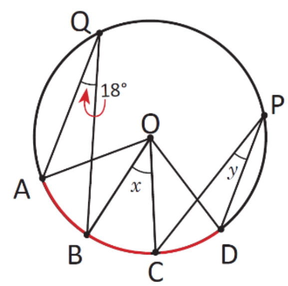
e) $\overset{\frown}{AB} = \overset{\frown}{CD}$



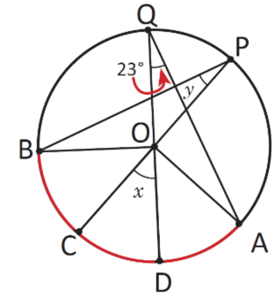
f) $\overset{\frown}{AB} = \overset{\frown}{BC}$



g) $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$

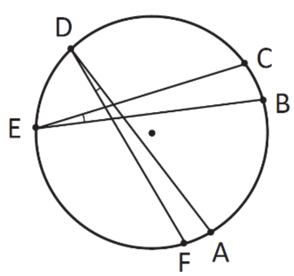


h) $\overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DA}$

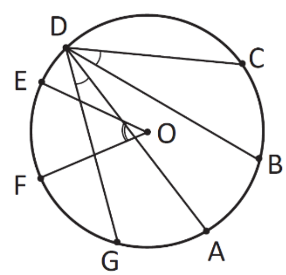


2. On the following circumferences, determine the arcs that are of equal measure.

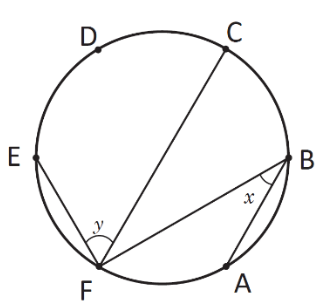
a) ∢ADF = ∢CEB



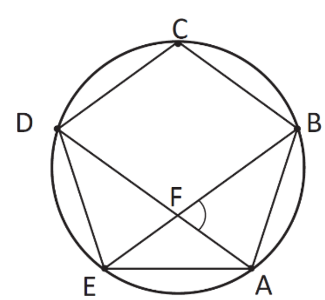
b) ∢FOE = 2∢CDB y ∢BDC = ∢ADG



3. Use the figure and determine the value of x and y if the points A, B, C, D, E, and F divide the circumference into six equal arcs.



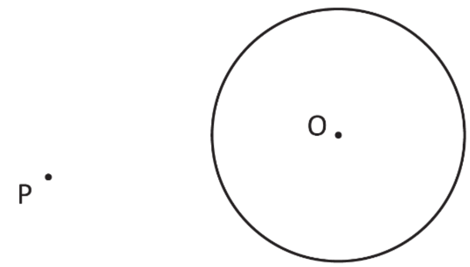
4. In the figure A, B, C, D and E is a regular pentagon, draw the diagonals AD and BE. Determine the measure of ∢BFA.



2.1 Construction of tangents to a circumference

P

Given the following circumference and the point marked as P, construct with a ruler and compass the lines that pass through point P and are tangent to the circumference.



S

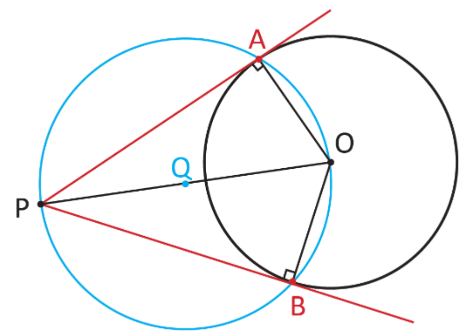
Taking the midpoint of the segment PO, denoted by Q.

Draw the circumference with the center Q and radius QO.

Draw dots for A and B where the circumference intersects.

Then, ∢OAP = ∢PBO = 90° (both subtend a 180° arc).

Therefore, the lines PA and PB are tangents to the circumference of center O.



The line perpendicular to the radius at a point in the circumference is the tangent to the circumference.

C

Using the inscribed angle results, one can construct the lines passing through a point P and tangent to a given circumference following the steps of the solution.

1. Draw a new circumference and P point outside the circumference, and construct the tangents to the circumference passing through the point P.

2. Based on the exercises in class, respond:

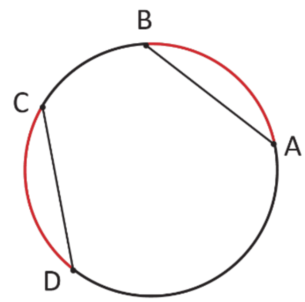
   a) Are PA and PB segments, the same?

   b) Why?

You can apply triangle congruence to justify your answer.

2.2 Chords and arcs of the circumference

In the following figure $\overset{\frown}{AB} = \overset{\frown}{CD}$. Compare the length of chords AB and CD.



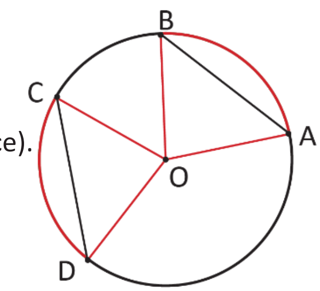
Draw the radiuses for OA, OB, OC and OD.

$\measuredangle BOA = \measuredangle DOC$ (because $\overset{\frown}{AB} = \overset{\frown}{CD}$).

OA = OB = OC = OD (are radiuses of the circumference).

Then, $\Delta BOA \cong \Delta DOC$ (as per SAS criterion).

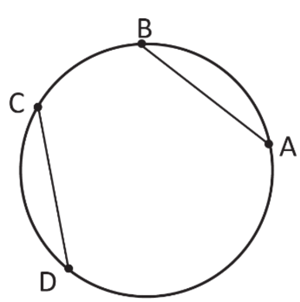
Therefore, AB = CD (per congruence).



To apply the SAS congruence criterion, two sides and the angle between them must be congruent.

In a circumference if the measure of the two arcs is equal, then the measure of the chord that subtends those arcs is equal.

In the following figure AB = CD. Compare the length of $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ arcs.

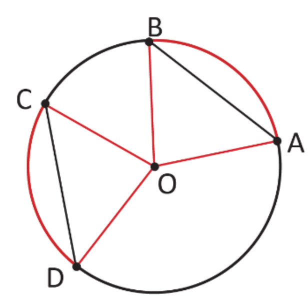


Draw the radiuses of OA, OB, OC and OD.

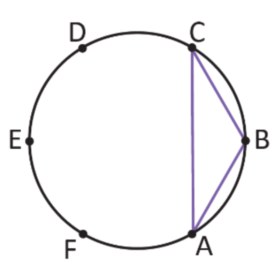
Then, $\Delta BOA \cong \Delta DOC$ (per LLL criterion).

Then , $\measuredangle BOA = \measuredangle DOC$ (per congruence).

Therefore, $\overset{\frown}{AB} = \overset{\frown}{CD}$ (the central angle is equal).



Points A, B, C, D, E, and F divide the circumference into six equal arcs. Clasify the figures formed by connecting the points indicated in each statement. Look at the example:



a) ABC BA = BC (because $\overset{\frown}{BA} = \overset{\frown}{BC}$).
R. ABC is an isosceles triangle.

b) ABDE

c) ACE

d) ACD

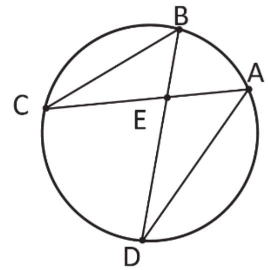
e) ABCDEF

f) DEF

g) ABCD

2.3 Similar triangle application

Determine if the following figure satisfies ΔAED ~ ΔBEC.

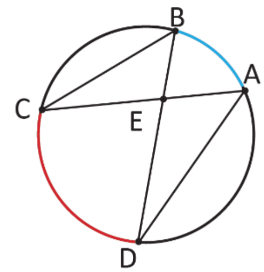


In the figure ∢AED = ∢BEC (are opposite by the vertex).

∢DBC = ∢DAC (subtend the same arc).

But ∢EBC = ∢DBC and ∢DAE = ∢DAC.

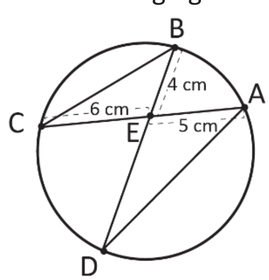
Therefore, ΔAED ~ ΔBEC (per criterion AA).



To apply the AA criterion it is only necessary two to be congruent.

It is necessary to observe the inscribed angles that subtend the same arc to determine the similarity between triangles.

The following figure determines the measure of the ED segment.



AS ΔAED ~ ΔBEC .

Then, $\frac{ED}{EC} = \frac{AE}{BE}$.

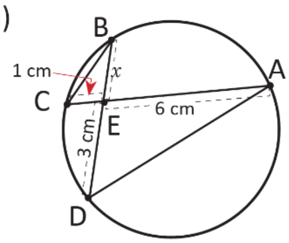
Therefore, $ED = EC \times \frac{AE}{BE} = 6 \times \frac{5}{4} = 7.5$.

ED = 7.5 cm

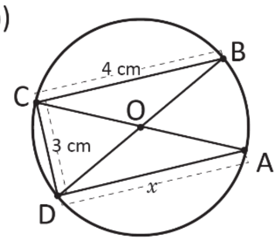
When two triangles are alike, the ratio between their homologous sides remains constant.

1. Determine x in the following figures:

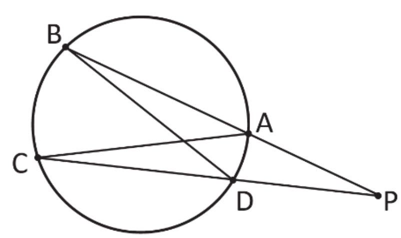
a)



b)



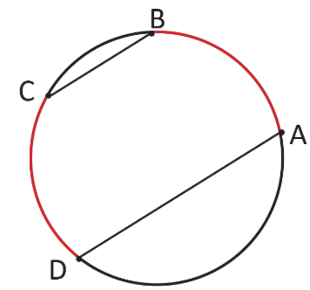
2. In the following figures determine what conditions are required for ΔACP ~ ΔDPB.



Is something else necessary?

2.4 Parallelism

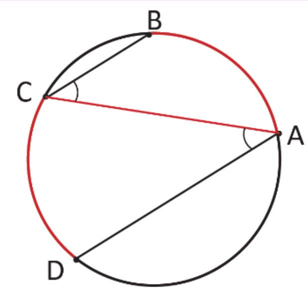
In the figure $\overset{\frown}{AB} = \overset{\frown}{CD}$. Determine if the segments AD and BC are parallels or secants.



Draw the chord AC.

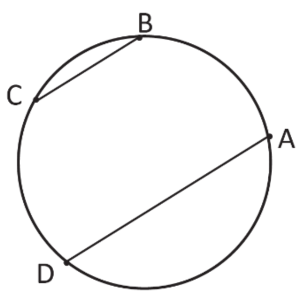
Then, ∢BCA = ∢DAC (since $\overset{\frown}{AB} = \overset{\frown}{CD}$).

Therefore, BC ∥ AD (alternate interior angles are equal).



If having two arcs of equal measure on a circumference, then the chords determined by the beginning of one arc and the end of the other are parallel.

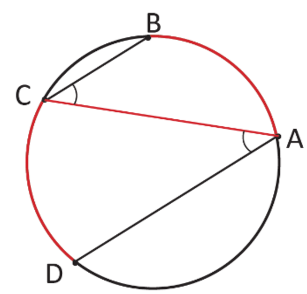
Compare the arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ from the circumference, if BC ∥ AD.



Draw the chord AC.

∢BCA = ∢DAC (alternate interior angles).

Therefore, $\overset{\frown}{AB} = \overset{\frown}{CD}$ (incribed angle theorem).



This result is reciprocal to the initial exercise.

Determine which of the following statements, are sufficient conditions to four consecutive points A, B, C, and D; on a circumference. Once connected, there is at least a pair of parallel chords.

a) $\overset{\frown}{AC} = \overset{\frown}{AD}$

b) ∢DBC = ∢BDA

c) CB = DA

d) $\overset{\frown}{CB} = \overset{\frown}{AD}$

e) AB = BC

f) ∢ACD = ∢ADB

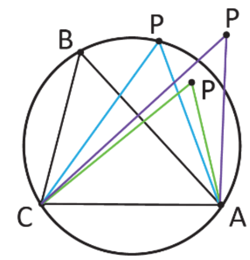
g) AC = BD

h) ΔABC ≅ ΔDCB

2.5 Four points on a circumference of a circle

P

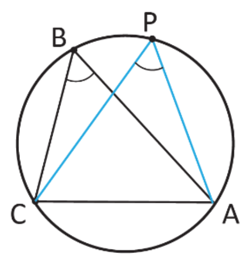
Considering ∢ABC = ∢APC and that both angles share the AC segment.
It shows that A, B, C and P are on the same circumference.



S

Point P has three options; on, in or out of the circumference.

Option 1

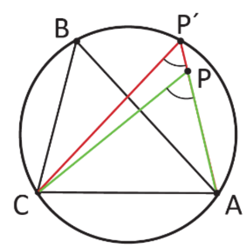


In this case:

∢ABC = ∢APC.

Therefore, A, B, C and P should stay in the same circumference.

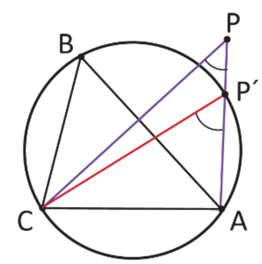
Option 2



Drawing ∢AP'C, then
∢ABC = ∢AP'C < ∢APC
Since ∢APC = ∢AP'C + ∢P'CP

Therefore, **∢ABC < ∢APC**.

Option 3



Drawing ∢AP'C, then
∢ABC = ∢AP'C > ∢APC.
Since ∢AP´C = ∢APC + ∢PCP'.

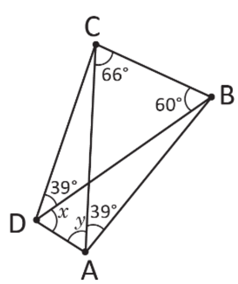
Therefore, **∢ABC > ∢APC**.

C

If two equal angles also share a segment at their openings, then the four points are on the same circumference.

E

Determine the value x and y.

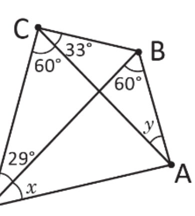


Since ∢CAB = ∢CDB and both share the CB, then A, B, C, D are on the same circumference.

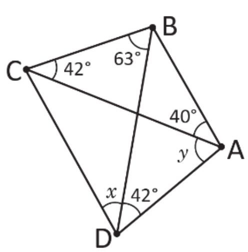
It must satisfied that **∢BCA = ∢BDA**, then x = 66°.
Moreover, it must meet that **∢CBD = ∢CAD**, then y = 60°.

Determine the value of x and y.

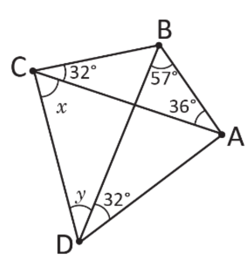
a)



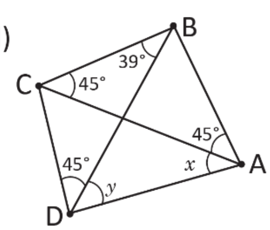
b)



c)



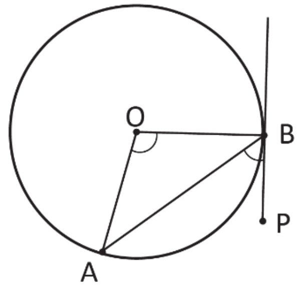
d)



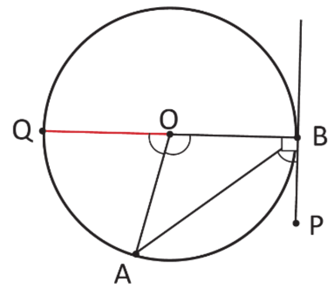
2.6 Semi-Inscribed angle

P

Compare the measurement of ∢ABP with ∢BOA in the following figure:



S



Draw the diameter for QB.

Then, ∢AOQ = 2∢ABO (Incribed Angle Theorem).

So, ∢AOQ = 180° - ∢BOA (Supplementary angle).

Then 2 ∢ABO = 180° - ∢BOA, therefore, ∢ABO = 90° - $\frac{∢BOA}{2}$.

Therefore, ∢PBA = $\frac{∢BOA}{2}$, or ∢BOA = 2∢PBA (per complementary angle, since PB ⊥ BO).

C

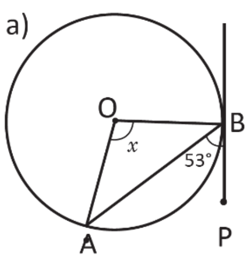
The angle formed by a tangent and a chord of the circumference is called: **semi-inscribed angle**.

In a circumference, **the measurement of a semi-inscribed angle is equal to half the measurement of the central angle, which subtends the same arc as the chord.**

E

Determine the value of x for each case:

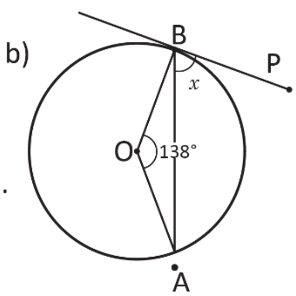
a)



As ∢BOA = 2∢PBA.

Therefore, x = 2(53°) = 106°.

b)

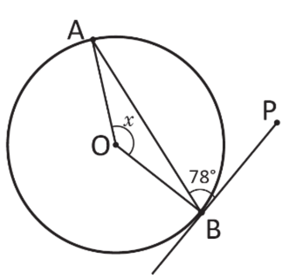


As ∢PBA = $\frac{1}{2}$ ∢BOA.

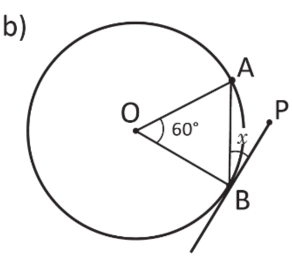
Therefore, $x = \frac{138°}{2}$ = 69°.

Determine the value of x for each case:

a)



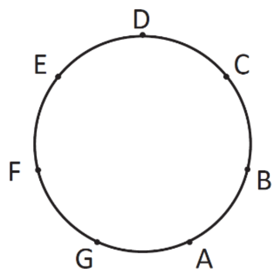
b)



2.7 Practice what you learned

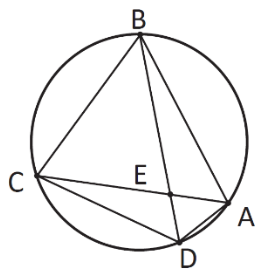
1. Draw a circumference and a dot on the outside of it. Use a ruler and a compass to draw the tangents across P.

2. Dots A, B, C, D, E, F, G divide the circumference into seven equal arcs. Classify the figures formed by connecting the dots indicated in each statement.



a) ABC b) ACDF c) ADG d) ABCDEFG

3. The following figures A, B, C, D are in the circumference. Respond:



a) What are the angles ∢EAB and ∢EDC?

b) What are the angles ∢ABE and ∢ACD? Why?

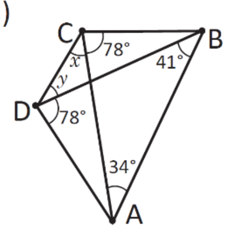
c) What are the angles ΔABE and ΔDCE? Why?

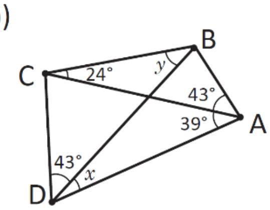
4. Determine which of the following statements are sufficient conditions to four consecutive points A, B, C, D on a circumference. Once connected, there is at least a pair of parallel chords.

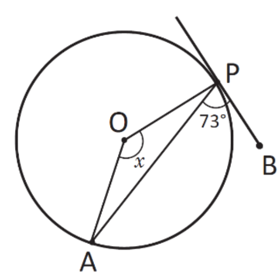
a) $\widehat{AC} = \widehat{BD}$ b) ∢CAB = ∢CDB c) AC = AD d) ΔABC ~ ΔCDA

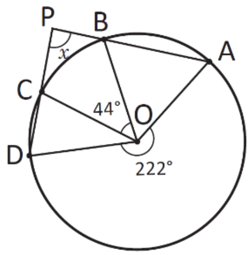
2.8 Practice what you learned

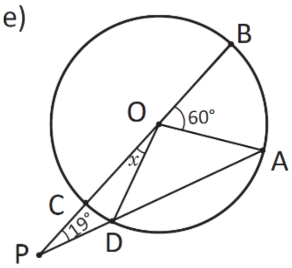
Determine the value of x or y, accordingly:

a)


b)


c)


d)


e)


f)